Title	The Notion and Aspects of Quantum Integrability(6) Approaches from mathematical science and quantum information, Chaos and Nonlinear Dynamics in Quantum-Mechanical and Macroscopic Systems)
Author(s)	Robnik, Marko
Citation	物性研究 (2005), 84(3): 541-545
Issue Date	2005-06-20
URL	http://hdl.handle.net/2433/110186
Right	
Type	Departmental Bulletin Paper
Textversion	publisher

The Notion and Aspects of Quantum Integrability

Marko Robnik

CAMTP - Center for Applied Mathematics and Theoretical Physics, University of Maribor, Krekova 2, SI-2000 Maribor, Slovenia

E-mail: Robnik@uni-mb.si

Abstract. In this paper I briefly discuss the notion and aspects of quantum integrability.

1. Introduction

Integrability in the classical Hamilton dynamical systems is a very precisely defined notion, whilst in quantum mechanics we are still not sure as for how to define the integrability, and to what extent does it correspond to the classical integrability, and furthermore, how abundant is the property of quantum integrability in Hamilton systems and what happens when it is broken: Do we face some kind of quantum analogy of the KAM Theorem then? In this paper I shall briefly discuss these questions.

2. Classical integrability in Hamilton systems

A classical Hamilton system $H(\mathbf{q},\mathbf{p})$ with N freedoms is defined *integrable* if in $2N$-dim classical phase space $(\mathbf{q},\mathbf{p})$ there exist N constants of motion A_j, $1 \le j \le N$, $A_1 = H$, which are global, sufficiently smooth (analytical), functionally independent and pairwise in involution, s.t. all Poisson brackets vanish $\{A_i, A_j\}_P = 0$. Classical integrability has the important consequence that the classical phase space is decomposed into the invariant N-dim tori, which fill the phase space everywhere, except for the separatrices, where the period of motion is infinite. It is due to this topological and geometrical structure that there exist most natural generalized variables in phase space, namely actions $\mathbf{I}$ and angles $\mathbf{\Theta}$. After such a construction the functional form of $H = K(\mathbf{I})$ is unique and the angles $\mathbf{\Theta}$ are cyclic variables (they do not appear in the Hamilton function). There is another canonical transformation mapping $(\mathbf{\Theta},\mathbf{I})$ uniquely onto $(\mathbf{Q},\mathbf{P})$, such that each $I_j = h_j = Q_j^2 + P_j^2$, $j = 1,\ldots,N$, is a simple 1-dim harmonic oscillator, and we can write $H = K(h_1,\ldots,h_N)$, where K is a unique function.

3. Hamiltonians as functions of other Hamiltonians: Classically

For any $H(\mathbf{q},\mathbf{p})$ and $\tilde{H} = f(H)$, where f is a sufficiently smooth but otherwise arbitrary function, $\tilde{H}$ is topologically (its invariant sets) and dynamically equivalent to H, except for the change of time by the factor df/dH taken at constant energy $E = H$. More generally, it can be shown (Robnik 2005), that when considering sufficiently smooth but otherwise arbitrary functions $f(H_1, H_2, \ldots, H_m)$ of m Hamiltonians H_j, $1 \le j \le m$, nothing new is generated, because the enlarged phase space is exactly equivalent to the product phase spaces of the constituents H_j, so the dynamics is completely preserved, and we face the case of most general separability. At any fixed and conserved value of H_j's the motion in $\tilde{H} = f(H_1, H_2, \ldots, H_m)$ is the same as in H_j, except for the *constant* scaling factor of time. Therefore, classically, considering Hamiltonians as functions of other Hamiltonians gives nothing new. In case of classical integrability, as we have seen above, the Hamiltonian H can be written as a unique function of N 1-dim harmonic oscillators, $H = K(h_1, \ldots, h_N)$.

4. Hamiltonians as functions of other Hamiltonians: Quantally

Quantum mechanically Hamiltonians as functions of other Hamiltonians are more interesting than classically: $\hat{H}$ and $\hat{\tilde{H}} = f(\hat{H})$, where again f is arbitrary sufficiently smooth function, have the same basis in Hilbert space ψ_j, $j = 1, 2, \ldots$, but different spectra: For all j we have

$$\hat{H}\psi_j = E_j\psi_j \tag{1}$$

and

$$\hat{\tilde{H}}\psi_j = f(\hat{H})\psi_j = f(E_j)\psi_j \tag{2}$$

Also, if $\hat{A}$ is any constant of motion of $\hat{H}$, s.t. by definition $[\hat{A}, \hat{H}] = 0$, then it is also constant of motion of $\hat{\tilde{H}} = f(\hat{H})$. Therefore the quantum dynamics is also preserved under the transformation f.

5. Purely discrete spectra: Interpolation problem and isospectrality

There is an interpolation theorem in mathematics (Rudin 1970) which guarantees that the following statement about the purely discrete spectra holds true, including any *finite* degeneracies, but assuming that there are no accummulation points: Given any $\hat{H}$ with the spectrum E_j, and given the 1-dim harmonic oscillator Hamiltonian $\hat{h}_1$ with the spectrum e_j, there exists a sufficiently smooth function f s.t. $E_j = f(e_j)$, so that we have the isospectrality of $\hat{H}$ and $\hat{\tilde{H}}$, i.e. spec$\hat{H}$ = spec$\hat{\tilde{H}}$ = spec$f(\hat{h}_1)$. In fact, there are infinitely many suitable interpolation functions f which do the job. Of course, isospectrality of an arbitrary Hamiltonian $\hat{H}$ and $f(\hat{h}_1)$ does not yet imply integrability of $\hat{H}$ as we shall discuss below. Moreover, we can extend the statement to N-dim interpolation by looking for a function f s.t. $\hat{H}$ will have the same spectrum as $\hat{\tilde{H}} = f(\hat{h}_1, \hat{h}_2, \ldots, \hat{h}_N)$, where each $\hat{h}_j$ is a 1-dim harmonic oscillator in the same basis as $\hat{H}$. Here we have isospectrality of $\hat{H}$ and $\hat{\tilde{H}}$, and nonuniqueness of f, whilst for quantum integrability (see below) we must require the *equality = identity* $\hat{H} = \hat{\tilde{H}} = f(\hat{h}_1', \hat{h}_2', \ldots, \hat{h}_N')$, where $\hat{h}_j'$ are unitarily transformed $\hat{h}_j$, but then - contrary to the classical integrability - the functioin f is *not* unique.

The existence of interpolation function f and isospectrality of $\hat{H}$ and $f(\hat{h}_1)$ is particularly interesting when e.g. H is classically fully chaotic (e.g. an ergodic 2-dim billiard system), whilst $f(h_1)$ is of course fully integrable and even 1-dim system. This is a puzzle or even a paradox, as it might bring some confusion into the theory of spectral universality classes in quantum chaos (Robnik and Berry 1986, Robnik 1986, Crehan 1995): A given spectrum is at the same time energy spectrum of a classically integrable and of a classically chaotic system: So, which universality class applies (Casati et al 1980, Bohigas et al 1984, Guhr et al 1998, Robnik 1998, Robnik 2004), Poissonian or RMT? In this regard please see the discussion in (Robnik and Berry 1986). The situation can be clarified by the following three remarks:

(**R1**) The universality classes of spectral fluctuations are defined only and strictly in the limit when the effective Planck constant $\hbar$ goes to zero: In a given finite and small energy interval we must asymptotically collect infinitely many energy levels.

(**R2**) In the limit $\hbar \to 0$ the interpolation construction of f might fail, and I assume it typically does so, except if H itself is classically integrable (as then we do have the EBK torus quantization). In addition, if f is $\hbar$-independent, then in this limit it does not affect the (unfolded) spectra at all, as it becomes locally just a linear transformation (multiplication by a constant df/dE).

(**R3**) The Hamilton operator $\hat{\tilde{H}} = f(\hat{h}_1)$ is *nonlocal*, because in general the Taylor expansion of f will contain all derivatives of f and powers of $\hat{h}_1$ to all orders up to infinity. It differs fundamentally from the Hamiltonians of the type *quadratic kinetic energy + potential.*

Therefore, in the limit $\hbar \to 0$ there is no problem with the classification of universality classes of spectral fluctuations: We can clearly distinguish between them, and the classical limit is important.

6. Quantum integrability: A definition and consequences

We have seen that all quantum Hamilton systems with purely discrete energy spectrum might be integrable *in some sense*, as they are isospectral to integrable systems, e.g. isospectral to even just a simple function $f(\hat{h}_1)$ of the 1-dim harmonic oscillator $\hat{h}_1$! Clearly, we see that such f, although analytic function, is not unique, and even less so when N-dim interpolation is performed. This N-dim interpolation reminds us of the Birkhoff normal form in classical Hamiltonian systems, which is derived from the construction of action angle variables, and which is unique, as discussed in section 2.

Here I assume, that for any quantum Hamilton system H with purely discrete energy spectrum the following construction $\hat{H} = \hat{\tilde{H}} = f(\hat{h}'_1, \hat{h}'_2, \ldots, \hat{h}'_N)$ with the transformation function f always exists, but is not necessarily unique. Here each $\hat{h}'_j$ is unitarily transformed $\hat{h}_j$. In such case *the system is defined quantum integrable.* This is in deep analogy to the classical integrability (see section 2), except that the quantal construction (the function f) is not unique.

Furthermore, this is in agreement with my rather speculative arguments at the time

(Robnik 1986) that such a construction succeeds in systems with purely discrete energy spectrum, which can be supported by the structure of quantal perturbation theory.

If this assumption is correct, it implies that (almost) all quantum Hamilton systems with purely discrete spectrum are integrable, but the classical limit of the constants of motion does not exist, because the transformation f fails when $\hbar \to 0$, except if the underlying Hamiltonian is classically integrable, in which case f becomes unique and equal to the classical K function. This is perfectly consistent with our picture of classical (non)integrability. If this statement is true, then the quantum analogy of the classical KAM Theorem is simply that quantum integrable systems with purely discrete spectrum remain integrable under a typical (generic) perturbation which preserves the discreteness of the spectrum.

7. More about quantum integrability

For quantum integrability we need more than isospectrality of two operators, we require identity. Let us first consider a formulation suitable mainly for one degree of freedom systems, $N = 1$. Suppose $\hat{H}$ is a given Hamilton operator with the spectrum E_j and orthonormal eigenbasis ψ_j, and $\hat{h}$ is another one with the spectrum e_k and orthonormal eigenbasis ϕ_k. We are looking for a function f such that not only the spectra E_j and $f(e_j)$ are identical, but that the operators $\hat{H}$ and $f(\hat{h})$ are identical, which means that

$$\hat{H}\psi_j = f(\hat{h})\psi_j = E_j\psi_j, \tag{3}$$

for all $j = 1, \ldots$. The unitary basis transformation is equal to $A_{jk} =< \psi_j|\phi_k >$, so that $\psi_j = \sum_k A_{jk}\phi_k$. In order to satisfy identity (3) we must have the identity

$$\sum_l A_{jl} f(e_l)\phi_l = E_j\psi_j, \tag{4}$$

for all j. Multiplying scalarly this equation by ϕ_k from the right, we obtain the identity

$$A_{jk} f(e_k) = A_{jk} E_j \tag{5}$$

for all j, k. After a while of thinking (Robnik 2005) one realizes that the only possibility is to have the identity transformation $A_{jk} = \delta_{jk}$, or any other matrix which is obtained from the identity δ_{jk} by permutation of rows (or columns). We conclude: $\hat{H}$ can be identical to a function $f(\hat{h})$ of another $\hat{h}$ only if they have the same (orthonormal) eigenbasis, i.e. if they commute. In such case there is an infinity of possibilities to choose an analytic f. Here we see again the conclusion of Johann von Neumann (1981), that for any set of commuting operators with purely discrete spectrum all of them can be written as functions of another single operator $\hat{R}$ in the same basis, and thus nothing new regarding quantum integrability can be achieved in this way.

In order to recover a sensible definition of quantum integrability of section 6, we have to incorporate properly the quantum analogy of the classical canonical transformation $(\mathbf{q}, \mathbf{p}) \to (\mathbf{Q}, \mathbf{P})$, and that is the unitary transformation $\hat{U}$ in the Hilbert space, where

$\hat{U}$ is the unitary basis transformation such that $\psi_j = \hat{U}\phi_j$ for all j, and E_j and e_j are eigenvalues of $\hat{H}$ and $\hat{h}$, respectively. We want to have the identity of $\hat{H}$ and the function f of the unitarily transformed $\hat{h}$, namely $\hat{g} = f(\hat{U}\hat{h}\hat{U}^{-1})$. Indeed, $\hat{g}$ and $f(\hat{h})$ have the same spectrum $f(e_k)$, and thus

$$\hat{H}\psi_j = \hat{g}\psi_j = f(\hat{U}\hat{h}\hat{U}^{-1})\psi_j = E_j\psi_j. \tag{6}$$

Now, if we have the identity $f(e_j) = E_j$, with an analytic f, then our construction succeeds. This is of course always possible, because $\hat{U}$ always exists and f as well (Rudin 1970), although the latter one is not unique. In case of $N = 1$ we have thus transformed a given Hamiltonian $\hat{H}$ to the 1-dim harmonic oscillator $\hat{h}$. This property deserves the name *quantum integrability.* It is much less exceptional than classical integrability, in fact we expect that all quantal Hamiltonians with purely discrete spectrum are quantum integrable. The details of this picture for higher degrees of freedom $N > 1$ cannot be discussed here due to the lack of space, but will be published elsewhere (Robnik 2005).

Acknowledgements

This work was supported by the Ministry of Higher Education, Science and Technology of the Republic of Slovenia and by the Nova Kreditna Banka Maribor. The author thanks the organizers of Kyoto Conference "Chaos and Nonlinear Dynamics in Quantum-Mechanical and Macroscopic Systems", Professors K. Nakamura, K. Takatsuka and I. Ohba, held at the Yukawa Institute in Kyoto, on 8-10 December 2004, for the kind invitation and great hospitality.

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